

Skew cyclic codes: Hamming distance and decoding algorithms¹

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- GLN, **Generating idempotents in ideal codes**, ACM Communications in Computer Algebra, Vol 48, No. 3, Issue 189, September 2014, ISSAC poster abstracts, pp. 113-115.
- GLN, **A new perspective of cyclicity in convolutional codes**, IEEE Transactions on Information Theory 62 (5) (2016), 2702–2706.
- GLN, **A Sugiyama-like decoding algorithm for convolutional codes**, 2016. arXiv:1607.07187.
- GLN, **Ideal codes over separable ring extensions**, IEEE Transactions on Information Theory 63 (5) (2017), 2796–2813.

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2 Skew Reed-Solomon codes

3 Decoding

Convolutional codes

\mathbb{F} a finite field, $k \leq n$ positive integers. A *rate k/n convolutional transducer* G transforms

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information sequences $\mathbf{u} = \dots \mathbf{u}_{-1} \mathbf{u}_0 \mathbf{u}_1 \dots$ ($\mathbf{u}_i \in \mathbb{F}^k$)

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Requirements: Write $\mathbf{u} = \sum_{i=-i_0}^{\infty} \mathbf{u}_i t^i \in \mathbb{F}^k((t)) \cong \mathbb{F}((t))^k$, $\mathbf{v} = \sum_{j=-j_0}^{\infty} \mathbf{v}_j t^j \in \mathbb{F}^n((t)) \cong \mathbb{F}((t))^n$.

Then

$$\mathbf{v} = \mathbf{u}G,$$

where G is an $k \times n$ full rank matrix with entries in $\mathbb{F}(t)$.

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Definition (Vector space version)

A rate k/n convolutional code over \mathbb{F} is a k -dimensional vector subspace of $\mathbb{F}(t)^n$.

Cyclic convolutional codes (module version)

Lemma

The map $\mathcal{D} \mapsto \mathcal{D} \cap \mathbb{F}^n[t]$ is a bijection between the set of rate k/n convolutional codes and the set of submodules of rank k of $\mathbb{F}^n[t] \cong \mathbb{F}[t]^n$ that are direct summands.

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This idea has been further developed, (e.g. considering more general finite \mathbb{F} -algebras A) by Roos (1979), Gluesing-Luerssen/Schmale (2004), López-Permouth/Szabo (2013), GLN (2014, 2017), among others.

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Main difficulty/opportunity: Dealing with idempotents in $A[t; \sigma]$.

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- Efficient decoding schemes (like Sugiyama, or Peterson-Gorenstein-Zierler) are adapted to this non-commutative setting.
- The theory (including the algorithms) work for any field L .

Skew cyclic codes

So, let L be a field, and $\sigma : L \rightarrow L$ a field automorphism of finite order n . Set $K = L^\sigma$. Consider the ring $\mathcal{R} = L[x; \sigma] / \langle x^n - 1 \rangle \cong M_n(K)$.

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We have the isomorphism of L -vector spaces $\mathfrak{p} : L^n \rightarrow \mathcal{R}$ sending $(c_0, c_1, \dots, c_{n-1})$ onto $c_0 + c_1x + \dots + c_{n-1}x^{n-1}$.

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Definition

A k -dimensional L -linear code $\mathcal{C} \subseteq L^n$ of dimension n is said to be a *skew cyclic code* if $\mathfrak{p}(\mathcal{C})$ is a left ideal of \mathcal{R} .

Note: We will identify \mathcal{C} with $\mathfrak{p}(\mathcal{C})$. Since every left ideal of \mathcal{R} is principal, we will often speak of the “skew code generated by a polynomial”, aggravating the abuse of language.

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Note: When $L = \mathbb{F}$ is a finite field, these codes lie in the realm of the theory developed by Boucher/Chaussade/Geiselmann/Loidreau/Ulmer (2007, 2009), among others, where the name is taken.

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Idea: Choosing carefully generator polynomials of skew codes, we get very concrete results.

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- For $1 \leq k < n$, the left ideal \mathcal{C} of \mathcal{R} generated by

$$g = [x - \beta, x - \sigma(\beta), x - \sigma^2(\beta), \dots, x - \sigma^{n-k-1}(\beta)]_\ell$$

is an SCC of length n and dimension k . We call them Reed-Solomon skew codes.

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Theorem

The Hamming minimum distance of \mathcal{C} is $\delta = n - k + 1$. Thus, it is an MDS code.

Parity check matrix

A parity check matrix of the skew RS code is given by

$$H = \begin{pmatrix} N_0(\beta) & N_0(\sigma(\beta)) & \dots & N_0(\sigma^{n-k-1}(\beta)) \\ N_1(\beta) & N_1(\sigma(\beta)) & \dots & N_1(\sigma^{n-k-1}(\beta)) \\ N_2(\beta) & N_2(\sigma(\beta)) & \dots & N_2(\sigma^{n-k-1}(\beta)) \\ \vdots & \vdots & \ddots & \vdots \\ N_{n-1}(\beta) & N_{n-1}(\sigma(\beta)) & \dots & N_{n-1}(\sigma^{n-k-1}(\beta)) \end{pmatrix}.$$

Here, for $\gamma \in L$,

$$N_j(\gamma) = \gamma\sigma(\gamma)\dots\sigma^{j-1}(\gamma)$$

is the remainder of the *left division* of x^j by $x - \gamma$, that is

$$x^j = q(x)(x - \gamma) + N_j(\gamma).$$

Transmission and syndromes

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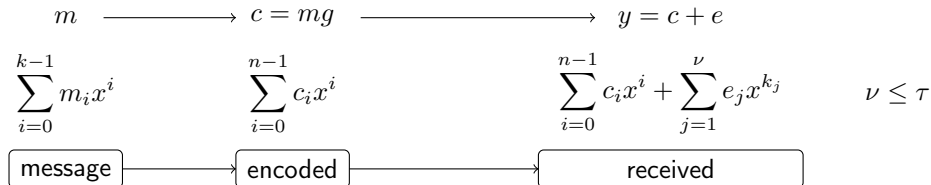
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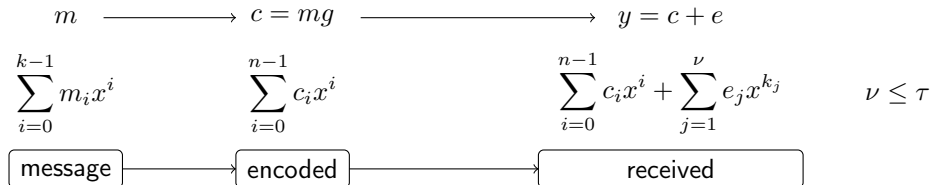


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From the received polynomial $y = \sum_{j=0}^{n-1} y_j x^j$ we can compute the *syndromes*

$$S_i = \sum_{j=0}^{n-1} y_j N_j(\sigma^i(\beta)), \quad i = 0, \dots, n-1$$

Error locator and error evaluator

We thus know the *syndrome polynomial*, defined as

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$$\lambda = [1 - \sigma^{k_1}(\beta)x, 1 - \sigma^{k_2}(\beta)x, \dots, 1 - \sigma^{k_\nu}(\beta)x]_r$$

For $1 \leq j \leq \nu$ we thus know that there is p_j of degree $\nu - 1$ such that $\lambda = (1 - \sigma^{k_j}(\beta)x)p_j$. Define the *evaluator polynomial*

$$\omega = \sum_{j=1}^{\nu} e_j \sigma^{k_j}(\alpha) p_j \tag{1}$$

Error locator and error evaluator

We thus know the *syndrome polynomial*, defined as

$$S = \sum_{i=0}^{2\tau-1} \sigma^i(\alpha) S_i x^i.$$

Define the *locator polynomial*

$$\lambda = [1 - \sigma^{k_1}(\beta)x, 1 - \sigma^{k_2}(\beta)x, \dots, 1 - \sigma^{k_\nu}(\beta)x]_r$$

For $1 \leq j \leq \nu$ we thus know that there is p_j of degree $\nu - 1$ such that $\lambda = (1 - \sigma^{k_j}(\beta)x)p_j$. Define the *evaluator polynomial*

$$\omega = \sum_{j=1}^{\nu} e_j \sigma^{k_j}(\alpha) p_j \quad (1)$$

Sugiyama's decoding scheme.

If λ is computed, then the error positions k_1, \dots, k_ν are derived. With these at hand, we can compute p_1, \dots, p_ν . Finally, the error values e_1, \dots, e_ν are computed by solving a linear system from (1), whenever ω is known.

The non-commutative key equation

Theorem

These polynomials satisfy the non-commutative key equation

$$\omega = S\lambda + x^{2\tau}u,$$

for some u of degree smaller than ν .

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Theorem

The non-commutative key equation

$$x^{2\tau}u + S\lambda = \omega \tag{2}$$

is a right multiple of the equation

$$x^{2\tau}u_I + Sv_I = r_I, \tag{3}$$

where u_I, v_I and r_I are the Bezout coefficients returned by the REEA with input $x^{2\tau}$ and S , and I is the index determined by the conditions $\deg r_{I-1} \geq \tau$ and $\deg r_I < \tau$. In particular, $\lambda = v_I g$ and $\omega = r_I g$ for some $g \in L[x; \sigma]$.

Sugiyama-like decoding

- If $(\lambda, \omega)_r = 1$, then the last theorem gives that the REEA serves for the computation of the locator polynomial λ and the evaluator polynomial ω .

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Sugiyama-like decoding

- If $(\lambda, \omega)_r = 1$, then the last theorem gives that the REEA serves for the computation of the locator polynomial λ and the evaluator polynomial ω .
- As mentioned above, once these polynomials are computed, the error polynomial is computed by a scheme similar to the Sugiyama algorithm for (commutative) RS codes.
- The condition $(\lambda, \omega)_r = 1$ is equivalent to

$$\deg v_I = \text{Cardinal}\{0 \leq i \leq n - 1 : 1 - \sigma^i(\beta)x \text{ left divides } v_I\}$$

Sugiyama-like decoding algorithm

Input: A polynomial $y = \sum_{i=0}^{n-1} y_i x^i$ received from the transmission of a codeword c in a skew RS code \mathcal{C} generated by $g = [\{x - \sigma^i(\beta)\}_{i=0, \dots, n-k-1}]_\ell$ of error-correcting capacity $\tau = \lfloor \frac{n-k}{2} \rfloor$.

Output: A codeword c' , or *key equation failure*.

- 1: **for** $0 \leq i \leq 2\tau - 1$ **do**
- 2: $S_i \leftarrow \sum_{j=0}^{n-1} y_j N_j(\sigma^i(\beta))$
- 3: $S \leftarrow \sum_{i=0}^{2\tau-1} \sigma^i(\alpha) S_i x^i$
- 4: **If** $S = 0$ **then**
- 5: **Return** y
- 6: $\{u_i, v_i, r_i\}_{i=0, \dots, l} \leftarrow \text{REEA}(x^{2\tau}, S)$
- 7: $I \leftarrow$ first iteration in REEA with $\deg r_i < \tau$, $pos \leftarrow \emptyset$
- 8: **for** $0 \leq i \leq n - 1$ **do**
- 9: **If** $1 - \sigma^i(\beta)x$ is a left divisor of v_I **then**
- 10: $pos = pos \cup \{i\}$
- 11: **If** $\deg v_I > \text{Cardinal}(pos)$ **then**
- 12: **Return** *key equation failure*
- 13: **for** $j \in pos$ **do**
- 14: $p_j \leftarrow \text{right-quotient}(v_I, 1 - \sigma^j(\beta)x)$
- 15: Solve the linear system $r_I = \sum_{j \in pos} e_j \sigma^j(\alpha) p_j$
- 16: $e \leftarrow \sum_{j \in pos} e_j x^j$
- 17: **Return** $y - e$

Key equation failures

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Theorem

The key equation failure happens if and only if e_1, \dots, e_ν are linearly dependent over $K = L^\sigma$.

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Theorem

The key equation failure happens if and only if e_1, \dots, e_ν are linearly dependent over $K = L^\sigma$.

On the other hand, we have an algorithm to solve the key equation failure, if we insist.